APPROXIMATION THEORY FOCUS GROUP - THE FIELDS INSTITUTE 2025 - PRESENTING H1 FOR COMPLETE FANS AND HYPERPLANE ARRANGEMENTS, WORKING NOTES.

MICHAEL DIPASQUALE AND NELLY VILLAMIZAR

The following lemma is a useful presentation for $H_1(\mathcal{J}^{\mathbf{r}}[\Sigma])$ when Σ is complete. This is the analogue of [5, Lemma 3.8] for complete fans.

Lemma 0.1 ([1, Lemma 9.12]). *Let $\Sigma \subset \mathbb{R}^3$ be a hereditary, complete fan. Define $K^r \subset \bigoplus_{\tau\in\Sigma_2} R(-\mathbf{r}(\tau)-1)$ by*

$$K^{\mathbf{r}} = \{\sum_{\tau\ni\gamma} f_\tau e_\tau | \gamma \in \Sigma_1, \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)+1} = 0\}.$$

Also define $V^r \subset \bigoplus_{\tau\in\Sigma_2} R(-\mathbf{r}(\tau)-1)$ by

$$V^{\mathbf{r}} = \{\sum_{\tau\in\Sigma_2} f_\tau e_\tau | \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)+1} = 0\}.$$

Then $K^{\mathbf{r}} \subset V^{\mathbf{r}}$ and $H_1(\mathcal{J}^{\mathbf{r}}[\Sigma]) \cong V^{\mathbf{r}}/K^{\mathbf{r}}$ as R-modules.

Proof. The proof is similar to the proof of [5, Lemma 3.8]. Let $K^{\mathbf{r}}_\gamma \subset \bigoplus_{\gamma\in\tau} R(-\mathbf{r}(\tau))e_\tau$ be the module of relations around the ray $\gamma \in \Sigma_1$, namely

$$K^{\mathbf{r}}_\gamma = \{\sum_{\tau\ni\gamma} f_\tau e_\tau | \sum f_\tau \alpha_\tau^{\mathbf{r}(\tau)} = 0\}.$$

Furthermore, let $J(\mathbf{0})$ be the ideal of the central vertex of Σ. Set up the following diagram with exact rows, whose first row is the complex $J[\Sigma]$.

$$\begin{tikzcd}
0 & 0 & 0 \\
\bigoplus\limits_{\tau\in\Sigma_2} J^{\mathbf{r}}(\tau) \arrow[r] \arrow[u] & \bigoplus\limits_{\gamma\in\Sigma_1} J^{\mathbf{r}}(\gamma) \arrow[r] \arrow[u] & J^{\mathbf{r}}(\mathbf{0}) \arrow[u] \\
\bigoplus\limits_{\tau\in\Sigma_2} R(-\mathbf{r}(\tau)) \arrow[r] \arrow[u] & \bigoplus\limits_{\substack{\gamma\in\Sigma_1,\tau\in\Sigma_2\\ \gamma\in\tau}} R(-\mathbf{r}(\tau)) \arrow[r] \arrow[u] & \bigoplus\limits_{\tau\in\Sigma_2} R(-\mathbf{r}(\tau)) \arrow[u] \\
0 \arrow[r] \arrow[u] & \bigoplus\limits_{\gamma\in\Sigma_1} K^{\mathbf{r}}_\gamma \arrow[r, "\iota"] \arrow[u] & V^{\mathbf{r}} \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

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The middle row is in fact exact because the inclusion on the left hand side has the effect of gluing together copies of $R(-\mathbf{r}(\tau))$ that correspond to different rays in Σ_1, leaving a copy of $R(-\mathbf{r}(\tau))$ for every codimension one face $\tau \in \Sigma_2$ in the cokernel. Now the long exact sequence in homology yields the isomorphisms $H_2(\mathcal{J}^{\mathbf{r}}[\Sigma]) \cong \ker(\iota)$ and $H_1(\mathcal{J}^{\mathbf{r}}[\Sigma]) \cong \operatorname{coker}(\iota)$. The image of $\bigoplus\limits_{\gamma\in\Sigma_1} K^{\mathbf{r}}_\gamma$ under ι is precisely $K^{\mathbf{r}}$, so we are done. □

Now suppose $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$. Let $\mathbf{r} : \Sigma_2^{\mathcal{A}} \to \mathbb{Z}_{\geq -1}$ be a smoothness distribution that is constant on hyperplanes (that is, if $\tau, \tau' \subset H \in \mathcal{A}$, then $\mathbf{r}(\tau) = \mathbf{r}(\tau')$). In this case, we also regard $\mathbf{r}$ as a map from $\mathcal{A} \to \mathbb{Z}_{\geq -1}$. Let $M^{\mathbf{r}} = \left[\alpha_1^{\mathbf{r}(H_1)} \cdots \alpha_k^{\mathbf{r}(H_k)}\right]$ be the matrix whose entries are the linear forms defining the hyperplanes of $\mathcal{A}$, raised to the power stipulated by $\mathbf{r}$. Let

$$\operatorname{syz}(M^{\mathbf{r}}) := \left\{ \sum_{i=1}^k f_i e_i : \sum_{i=1}^k f_i \alpha_i^{\mathbf{r}(H_i)} = 0 \right\} \subset \bigoplus_{i=1}^k R(-\mathbf{r}(H_i))$$

be the syzygy module of the matrix $M^{\mathbf{r}}$.

For a given line W appearing as the intersection of at least two hyperplanes of $\mathcal{A}$, we write $M^{\mathbf{r}}_W$ for the matrix with a single row whose entries are $\{\alpha_H^{\mathbf{r}(H)} : W \subset H\}$. We similarly have

$$\operatorname{syz}(M^{\mathbf{r}}_W) := \left\{ \sum_{H\supset W} f_H e_H : \sum_{H\supset W} f_H \alpha_H^{\mathbf{r}(H)} = 0 \right\} \subset \bigoplus_{H\supset W} R(-\mathbf{r}(H) - 1).$$

There is a natural inclusion from $\operatorname{syz}(M^{\mathbf{r}}_W)$ into $\operatorname{syz}(M^{\mathbf{r}})$ by extending the syzygy on $M^{\mathbf{r}}_W$ by zero to the rest of the entries of $\operatorname{syz}(M^{\mathbf{r}})$.

Corollary 0.2. *If $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$, and $\mathbf{r} : \Sigma_2^{\mathcal{A}} \to \mathbb{Z}_{\geq 0}$ is a smoothness distribution, then*

$$H_1(\mathcal{J}[\Sigma^{\mathcal{A}}]) \cong \frac{\operatorname{syz}(M^{\mathbf{r}})}{\sum_{W\in L_2(\mathcal{A})} \operatorname{syz}(M^{\mathbf{r}}_W)},$$

where $L_2(\mathcal{A})$ is the collection of lines appearing as intersections of hyperplanes of $\mathcal{A}$.

1. Three-generated hyperplane arrangements

The case $\mathbf{r} \equiv \mathbf{0}$ of Corollary 0.2 deserves special attention.

Corollary 1.1. *If $\mathcal{A} = \bigcup_{i=1}^k H_i \subset \mathbb{R}^3$ is a central and essential hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$ and $\mathbf{r} \equiv \mathbf{0}$, then $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])$ is isomorphic to the $\mathbb{R}$-vector space of $\mathbb{R}$-linear relations among the linear forms $\alpha_1, \ldots, \alpha_k$ modulo the $\mathbb{R}$-linear relations among $\alpha_1, \ldots, \alpha_k$ of length three.*

Proof. In this case, $\mathbf{r}(\tau) = 1$ for all $\tau \in \Sigma_2^{\mathcal{A}}$, so $M^{\mathbf{r}} = \left[\alpha_1 \cdots \alpha_k\right]$. Suppose $\alpha_1, \alpha_2, \alpha_3$ are a basis for the $\mathbb{R}$-span of the entries of $M^{\mathbf{r}}$ (this is three dimensional since $\mathcal{A}$ is essential).

Then $\operatorname{syz}(M^{\mathbf{r}})$ is generated by the Koszul syzygies on $\{\alpha_1, \alpha_2, \alpha_3\}$ along with all the $\mathbb{R}$-linear relations on the entries of $M^{\mathbf{r}}$.

If $\bar{\gamma} \in L_2(\mathcal{A})$, then we can select two linear forms, without loss suppose these are α_1 and α_2, that intersect in the line $\bar{\gamma}$. The syzygy module $\operatorname{syz}(M^{\mathbf{r}}_{\bar{\gamma}})$ is generated by the Koszul syzygy between α_1 and α_2, along with the $\mathbb{R}$-linear relations on $\{\alpha_H\}_{\gamma\in H}$. Since these linear forms effectively live in the two-dimensional vector space spanned by α_1 and α_2, the relations among them all have length three.

From the above descriptions, we see that the Koszul syzygies in $\mathrm{syz}(M^{\mathbf{r}})$ appear also in $\sum\limits_{\bar\gamma\in L_2(\mathcal{A})}\mathrm{syz}(M^{\mathbf{r}}_{\bar\gamma})$. Thus the presentation in Corollary 0.2 implies that

$$H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])\cong\frac{\mathrm{syz}_0(M^{\mathbf{r}})}{\sum_{\bar\gamma\in L_2(\mathcal{A})}\mathrm{syz}_0(M^{\mathbf{r}}_{\bar\gamma})},$$

where syz_0 represents 'syzygies of degree zero' – that is, $\mathbb{R}$-linear relations.

Furthermore, any relation of length three among $\{\alpha_1,\ldots,\alpha_k\}$, without loss suppose $c_1\alpha_1+c_2\alpha_2+c_3\alpha_3=0$, necessarily expresses the fact that α_1,α_2, and α_3 all vanish along a common line $\bar\gamma\in L_2(\mathcal{A})$. Thus this relation appears in $\mathrm{syz}(M^{\mathbf{r}}_{\bar\gamma})$. It follows that we may recast the above presentation as the space of all $\mathbb{R}$-linear relations on $\alpha_1,\ldots,\alpha_k$ modulo the space of $\mathbb{R}$-linear relations of length three. □

Definition 1.2. If $\mathcal{A}=\bigcup_{i=1}^k H_i$ is a hyperplane arrangement with H_i the vanishing locus of α_i for $i=1,\ldots,k$, then $\mathcal{A}$ is called 3-generated if the space of all $\mathbb{R}$-linear relations among $\alpha_1,\ldots,\alpha_k$ is generated by the relations of length 3.

Lemma 1.3. *If $\mathcal{A}=\bigcup_{i=1}^k H_i\subset\mathbb{R}^3$ is a hyperplane arrangement with associated complete fan $\Sigma^{\mathcal{A}}$, and $\mathbf{r}:\Sigma_2^{\mathcal{A}}\to\mathbb{Z}_{\ge 0}$ is a smoothness distribution, then $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])$ has finite length. Furthermore $S^{\mathbf{r}}(\Sigma^{\mathcal{A}})$ is free if and only if $H_1(\mathcal{J}[\Sigma^{\mathcal{A}}])=0$.*

Sketch of proof. Show that the localization of the presentation in Corollary 0.2 at all homogeneous prime ideals besides the maximal ideal vanishes. The latter fact (concerning freeness) follows from a seminal result of Schenck [3], generalized in [4, Theorem 3.4]. In the three-dimensional case, this can be argued fairly quickly using Ext. □

Corollary 1.4. *$S^0(\Sigma^{\mathcal{A}})$ is free if and only if $\mathcal{A}$ is 3-generated.*

The subtlety of this can be seen in action with an example that is sometimes called *Ziegler's pair*. There will be a Macaulay demo walking through this example.

2. Unwinding the dimension of splines on complete three-dimensional fans

We regard $\mathbf{r}$ and Σ as understood in this section, and so we write $\mathcal{J}$ instead of $\mathcal{J}^{\mathbf{r}}[\Sigma]$. If Σ is a complete three-dimensional fan, Lemma 0.1 can be used to give an interesting dimension formula for $S^{\mathbf{r}}(\Sigma)$. Using the Euler characteristic of $\mathcal{J}$ and the fact that $H_0(\mathcal{J})=0$ we get

$$\dim H_2(\mathcal{J})_d-\dim H_1(\mathcal{J})_d=\sum_{\tau\in\Sigma_2}\dim J(\tau)_d-\sum_{\gamma\in\Sigma_1}J(\gamma)_d+\dim J(\mathbf{0})_d$$

Now recall that $S^{\mathbf{r}}(\Sigma)\cong R\oplus H_2(\mathcal{J})$. This gives

$$\dim S^{\mathbf{r}}(\Sigma)_d=\dim R_d+\sum_{\tau\in\Sigma_2}\dim J(\tau)_d-\sum_{\gamma\in\Sigma_1}\dim J(\gamma)_d+\dim J(\mathbf{0})_d+\dim H_1(\mathcal{J})_d. \tag{2.1}$$

Since $R=\mathbb{R}[x,y,z]$, $\dim R_d=\binom{d+2}{2}$. Since $J(\tau)=\langle\alpha_\tau^{\mathbf{r}(\tau)+1}\rangle$, $\dim J(\tau)_d=\binom{d+2-(\mathbf{r}(\tau)+1)}{2}=\binom{d+1-\mathbf{r}(\tau)}{2}$. The dimension of $J(\gamma)_d$ is computed by Schenck and Geramita in [2]. Their result is the following. $J(\gamma)=\langle\alpha_\tau^{\mathbf{r}(\tau)+1}:\tau\ni\gamma\rangle$ is a codimension two Cohen-Macaulay ideal, and thus it has a Hilbert-Burch resolution. Suppose that there are $n_\gamma+1$ codimension one cones containing γ and that $\tau_0,\ldots,\tau_{n_\gamma}\in\Sigma_2$ are these codimension one cones. Then $J(\gamma)$ is generated in degrees $e_0=\mathbf{r}(\tau_0)+1,\cdots,e_{n_\gamma}=\mathbf{r}(\tau_{n_\gamma})+1$. Put $E_\gamma=\sum_{i=0}^{n_\gamma}e_i$. The key technical result of [2], which is Theorem 2.7 of that paper, is that $\mathrm{syz}J(\gamma)$ is generated in degrees $d_1,\ldots,d_{n_\gamma}$ where $d_1\le d_2\le\cdots\le d_{n_\gamma}$ are each as close as possible to E_γ/n_γ. To state this exactly, let $E_\gamma=q_\gamma n_\gamma+r_\gamma$

be the result of dividing E_γ by n_γ using the Euclidean algorithm. Then $d_1 = d_2 = \ldots = d_{n_\gamma - r_\gamma} = q_\gamma$ and $d_{n_\gamma - r_\gamma + 1} = \cdots = d_{n_\gamma} = q_\gamma + 1$. Thus

$$\dim J(\gamma)_d = \sum_{\tau \ni \gamma} \binom{d+1-\mathbf{r}(\tau)}{2} - (n_\gamma - r_\gamma)\binom{d+2-q_\gamma}{2} - r_\gamma \binom{d+1-q_\gamma}{2}.$$

In Equation (2.1), this leaves the term $\dim J(0)_d + \dim H_1(\mathcal{J})_d$ unsimplified. We simplify it as follows. Consider the (generally non-minimal) exact sequence

$$0 \to \mathrm{syz}[\alpha_\tau^{\mathbf{r}(\tau)+1} : \tau \in \Sigma_2] \to \bigoplus_{\tau \in \Sigma_2} R(-\mathbf{r}(\tau) - 1) \xrightarrow{[\alpha_\tau^{\mathbf{r}(\tau)+1} : \tau \in \Sigma_2]} J(0) \to 0.$$

In the notation of Lemma 0.1, $\mathrm{syz}[\alpha_\tau^{\mathbf{r}(\tau)+1} : \tau \in \Sigma_2] = V^{\mathbf{r}}$. Thus we have $\dim J(0) = \sum_{\tau \in \Sigma_2} \binom{d+1-\mathbf{r}(\tau)}{2} - \dim V^{\mathbf{r}}_d$. By Lemma 0.1, we have $\dim H_1(\mathcal{J})_d = \dim V^{\mathbf{r}}_d - \dim K^{\mathbf{r}}_d$. Thus

$$\begin{aligned}\dim J(0)_d + \dim H_1(\mathcal{J})_d =& \left(\sum_{\tau \in \Sigma_2} \binom{d+1-\mathbf{r}(\tau)}{2} - \dim V^{\mathbf{r}}_d\right) + (\dim V^{\mathbf{r}}_d - \dim K^{\mathbf{r}}_d) \\ =& \sum_{\tau \in \Sigma_2} \binom{d+1-\mathbf{r}(\tau)}{2} - \dim K^{\mathbf{r}}_d.\end{aligned}$$

Using these expressions to simplify (2.1) and using the fact that $\sum\limits_{\tau \in \Sigma_2} \sum\limits_{\tau \ni \gamma} \binom{d+1-\mathbf{r}(\tau)}{2} = 2 \sum\limits_{\tau \in \Sigma_2} \binom{d+1-\mathbf{r}(\tau)}{2}$ (each codimension one cone τ contains two rays) we get the following.

Theorem 2.1.

$$\dim S^{\mathbf{r}}(\Sigma)_d = \binom{d+2}{2} + \sum_{\gamma \in \Sigma_1} \left((n_\gamma - r_\gamma)\binom{d+2-q_\gamma}{2} - r_\gamma \binom{d+1-q_\gamma}{2} \right) - \dim K^{\mathbf{r}}_d.$$

Using Corollary 0.2, there is a different formulation of this theorem for a fan $\Sigma^{\mathcal{A}}$ induced by a hyperplane arrangement.

Theorem 2.2. *Let $\mathcal{A}$ be a central hyperplane arrangement and $\Sigma = \Sigma^{\mathcal{A}}$ its induced fan. Suppose that $\mathbf{r} : \Sigma_2 \to \mathbb{Z}_{\geq -1}$ is constant along hyperplanes of $\mathcal{A}$ (so we may regard $\mathbf{r}$ also as a function $\mathbf{r} : \mathcal{A} \to \mathbb{Z}_{\geq -1}$). Then*

$$\begin{aligned}\dim S^{\mathbf{r}}(\Sigma) = \binom{d+2}{2} - \sum_{\tau \in \Sigma_2} \binom{d+1-\mathbf{r}(\tau)}{2} + \sum_{\gamma \in \Sigma_1} \left((n_\gamma - r_\gamma)\binom{d+2-q_\gamma}{2} - r_\gamma \binom{d+1-q_\gamma}{2} \right) \\ + \sum_{H \in \mathcal{A}} \binom{d+1-\mathbf{r}(H)}{2} - \dim \left(\sum_{W \in L_2(\mathcal{A})} \mathrm{syz}(M^{\mathbf{r}}_W) \right)\end{aligned}$$

3. Generic Hyperplane arrangements

A hyperplane arrangement $\mathcal{A} = \cup_{i=1}^k H_i \subset \mathbb{R}^3$ is called *generic* if no three distinct hyperplanes $H_i, H_j, H_k \in \mathcal{A}$ intersect along a common line. In this case we have the following explicit formulas.

If $\gamma \in \Sigma_1^{\mathcal{A}}$, then there are exactly four two-dimensional cones of $\Sigma^{\mathcal{A}}$ that contain γ, and these four span only two distinct planes $H, H' \in \mathcal{A}$ by the assumption that $\mathcal{A}$ is generic. So $J(\gamma) = \langle \alpha_H^{\mathbf{r}(H)+1}, \alpha_{H'}^{\mathbf{r}(H')+1} \rangle$ is a complete intersection.

Moreover $\mathrm{syz} M^{\mathbf{r}}_W$ is quite simple: each $W \in L_2(\mathcal{A})$ is the intersection of exactly two hyperplanes $H, H' \in \mathcal{A}$. Thus $\mathrm{syz} M^{\mathbf{r}}_W$ consists of the Koszul syzygy between $\alpha_H^{\mathbf{r}(H)+1}$ and $\alpha_{H'}^{\mathbf{r}(H')+1}$. It follows that $\sum_{W \in L_2} \mathrm{syz} M^{\mathbf{r}}_W$ is the image of the Koszul syzygies inside the free module $\bigoplus_{H \in \mathcal{A}} R(-\mathbf{r}(H) - 1)$.

Let us write $K_2(\mathcal{A},\mathbf{r})$ for the $|\mathcal{A}|\times\binom{|\mathcal{A}|}{2}$ matrix whose columns are the Koszul syzygies between all pairs $\alpha_H^{\mathbf{r}(H)+1},\alpha_{H'}^{\mathbf{r}(H')+1}$. That is, let $N=\bigoplus_{H\in\mathcal{A}} R(-\mathbf{r}(\tau)-1)$. Then $K_2(\mathcal{A},\mathbf{r})$ represents the map

$$\wedge^2 N \xrightarrow{K_2(\mathcal{A},\mathbf{r})} N.$$

We have the following proposition.

Proposition 3.1. *If $\mathcal{A}\subset\mathbb{R}^3$ is a generic hyperplane arrangement and $\Sigma=\Sigma^{\mathcal{A}}$ the induced fan, then*

$$\begin{aligned}\dim S^{\mathbf{r}}(\Sigma^{\mathcal{A}})_d =& \binom{d+2}{2}+\sum_{\tau\in\Sigma_2}\binom{d+1-\mathbf{r}(\tau)}{2}\\ &-\sum_{\substack{\gamma\in\Sigma_1\\ \gamma\subset H\cap H'}}\left[\binom{d+1-\mathbf{r}(H)}{2}+\binom{d+1-\mathbf{r}(H')}{2}-\binom{d-\mathbf{r}(H)-\mathbf{r}(H')}{2}\right]\\ &+\sum_{H\in\mathcal{A}}\binom{d+1-\mathbf{r}(H)}{2}-\dim(\mathrm{im}K_2(\mathcal{A},\mathbf{r}))_d\end{aligned}$$

This reduces finding $\dim S^{\mathbf{r}}(\Sigma^{\mathcal{A}})$ to finding the Hilbert function of the image of the Koszul matrix $K_2(\mathcal{A},\mathbf{r})$.

Example 3.2. Suppose $\mathcal{A}=\cup_{i=1}^4 H_i$ is a generic three-dimensional arrangement and $\mathbf{r}(H_i)=0$ for $1\le i\le 4$. Let $\alpha_i=\alpha_{H_i}$ for $1\le i\le 4$. Then

$$K_2(\mathcal{A},\mathbf{r})=\begin{array}{c} \\ H_1\\ H_2\\ H_3\\ H_4\end{array}\begin{array}{c}\begin{array}{cccccc}12 & 13 & 14 & 23 & 24 & 34\end{array}\\ \begin{bmatrix}\alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 & 0\\ -\alpha_1 & 0 & 0 & \alpha_3 & \alpha_4 & 0\\ 0 & -\alpha_1 & 0 & -\alpha_2 & 0 & \alpha_4\\ 0 & 0 & -\alpha_1 & 0 & -\alpha_2 & -\alpha_3\end{bmatrix}\end{array}.$$

References

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